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Department of Mathematics

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# A Delayed Option Pricing Formula (University of Manchester Probability Seminar)

Salah-Eldin A. Mohammed

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# A Delayed Option Pricing Formula

Salah Mohammed

<http://sfde.math.siu.edu/>

November 14, 2007

Probability Seminar

The University

Manchester, UK

- Joint work with M. Arriojas, Y. Hu and G. Pap.

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- Research supported by NSF Grants DMS-9975462 and DMS-0203368.

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- Hedging strategy.

# Introduction

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*Proposal*: Allow volatility to depend on the **history of the stock price**: Predictions about the evolution of financial variables take into account the knowledge of their past.

*Objective*: To derive an option pricing formula under stock-dynamics with finite memory. (Theorem 4).

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*European call options* can only be exercised at the maturity date.

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Consider a stock whose price  $S(t)$  at any time  $t$  satisfies the following **stochastic delay differential equation (sdde)**:

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on a probability space  $(\Omega, \mathcal{F}, P)$  with a filtration  $(\mathcal{F}_t)_{0 \leq t \leq T}$  satisfying the usual conditions.

# Delayed Stock Model-contd

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$C([-L, 0], \mathbf{R}) :=$  Banach space of continuous functions  $[-L, 0] \rightarrow \mathbf{R}$  given the sup norm.

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**Brownian motion**:  $W$ -one-dimensional standard, adapted to  $(\mathcal{F}_t)_{0 \leq t \leq T}$ .

# Feasibility of Delayed Stock Model

---

Model is **feasible**: Admits pathwise unique solution such that  $S(t) > 0$  almost surely for all  $t \geq 0$  whenever the initial path  $\varphi(t) > 0$  for all  $t \in [-L, 0]$ .

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- (ii)  $g : \mathbf{R} \rightarrow \mathbf{R}$  is continuous.
- (iii) Delays  $a$  and  $b$  are positive and fixed.

# Theorem 1

*Assume Hypotheses (E). Then the delayed stock model*

$$\left. \begin{aligned} dS(t) &= h(t, S(t-a))S(t) dt + g(S(t-b))S(t) dW(t), \\ &\quad t \in [0, T] \\ S(t) &= \varphi(t), \quad t \in [-L, 0] \end{aligned} \right\} \quad (1)$$

*admits a pathwise unique solution  $S$  for a given  $\mathcal{F}_0$ -measurable initial process  $\varphi : \Omega \rightarrow C([-L, 0], \mathbf{R})$ . If  $\varphi(0) > 0$  a.s., then  $S(t) > 0$  a.s. for all  $t \geq 0$ .*



# Proof of Theorem 1

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Define **minimum delay**  $l := \min\{a, b\} > 0$ .

Let  $t \in [0, l]$ . The delayed stock model gives

$$\left. \begin{aligned} dS(t) &= S(t)[h(t, \varphi(t - a)) dt + g(\varphi(t - b)) dW(t)] \\ t &\in [0, l] \\ S(0) &= \varphi(0). \end{aligned} \right\} \quad (1)$$

# Proof of Theorem 1– Cont'd

---

Define the semimartingale

$$N(t) := \int_0^t h(u, \varphi(u - a)) du + \int_0^t g(\varphi(u - b)) dW(u),$$

for  $t \in [0, l]$ .

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$$[N, N](t) = \int_0^t g(\varphi(u - b))^2 du, \quad t \in [0, l].$$

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Then (1) becomes

$$dS(t) = S(t) dN(t), \quad t > 0, \quad S(0) = \varphi(0),$$

with the unique solution:

# Proof of Theorem 1– Cont'd

$$\begin{aligned} S(t) &= \varphi(0) \exp\left\{N(t) - \frac{1}{2}[N, N](t)\right\}, \\ &= \varphi(0) \exp\left\{\int_0^t h(u, \varphi(u-a)) du \right. \\ &\quad \left. + \int_0^t g(\varphi(u-b)) dW(u) \right. \\ &\quad \left. - \frac{1}{2} \int_0^t g(\varphi(u-b))^2 du\right\}, \end{aligned}$$

for  $t \in [0, l]$ . This implies that  $S(t) > 0$  almost surely for all  $t \in [0, l]$ , when  $\varphi(0) > 0$  a.s..

# Proof of Theorem 1– Cont'd

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Similarly, since  $S(l) > 0$ , then  $S(t) > 0$  for all  $t \in [l, 2l]$  a.s.. Therefore  $S(t) > 0$  for all  $t \geq 0$  a.s., by induction using forward steps of lengths  $l$ .

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Above argument also gives **existence** and **pathwise uniqueness** of the strong solution to the delayed stock model.  $\diamond$



# Remark 1

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In the delayed stock model, we need only require  $\varphi(0) \geq 0$  (or  $\varphi(0) > 0$ ) to conclude that a.s.  $S(t) \geq 0$  for **all**  $t \geq 0$  (or  $S(t) > 0$  for **all**  $t \geq 0$ , resp.).

# An Extension of the Model

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Another feasible model for the stock price is

$$dS(t) = f(t, S^{t-a})S(t) dt + g(S(t-b))S(t) dW(t), \\ t \in [0, T],$$

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where  $f : [0, T] \times C([-L, T], \mathbf{R}) \rightarrow \mathbf{R}$  is a continuous functional; and  $S^t \in C([-L, T], \mathbf{R})$ ,  $t \in [-L, T]$ , is defined by

$$S^t(s) := S(t \wedge s), \quad s \in [-L, T],$$

for  $S \in C([-L, T], \mathbf{R})$ .

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- Positive delays  $a, b$ ;  $h, g$  continuous.
- $g(v) \neq 0$  whenever  $v \neq 0$ .

# Delayed Market – Cont'd

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- Obtain an equivalent local martingale measure (via Girsanov's theorem).
- Establish completeness and no-arbitrage property of the market.
- Obtain a hedging strategy.

# Discounted Stock

Let

$$\tilde{S}(t) := \frac{S(t)}{B(t)} = e^{-rt} S(t), \quad t \in [0, T],$$

be the **discounted stock price**. Then by the product rule:

$$\begin{aligned} d\tilde{S}(t) &= e^{-rt} dS(t) + S(t)(-re^{-rt}) dt \\ &= \tilde{S}(t) \left[ \{h(t, S(t-a)) - r\} dt \right. \\ &\quad \left. + g(S(t-b)) dW(t) \right]. \end{aligned}$$

# Discounted Stock – Cont'd

---

Define

$$\begin{aligned}\widehat{S}(t) &:= \int_0^t \{h(u, S(u-a)) - r\} du \\ &\quad + \int_0^t g(S(u-b)) dW(u),\end{aligned}$$

for  $t \in [0, T]$ .

# Discounted Stock – Cont'd

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for  $t \in [0, T]$ .

Then

$$d\widetilde{S}(t) = \widetilde{S}(t) d\widehat{S}(t), \quad 0 < t < T. \quad (2)$$

## Discounted Stock – Cont'd

---

Since  $\tilde{S}(0) = \varphi(0)$ , then

$$\tilde{S}(t) = \varphi(0) + \int_0^t \tilde{S}(u) d\hat{S}(u), \quad t \in [0, T]. \quad (3)$$

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To establish an **equivalent local martingale measure**, recall **Girsanov's theorem**:

# Theorem 2 (Girsanov)

---



## Theorem 2 (Girsanov)

*Let  $W(t)$ ,  $t \in [0, T]$ , be a standard Wiener process on  $(\Omega, \mathcal{F}, P)$ . Let  $\Sigma$  be a predictable process such that  $\int_0^T |\Sigma(u)|^2 du < \infty$  a.s.. Define*

$$\varrho(t) := \exp \left\{ \int_0^t \Sigma(u) dW(u) - \frac{1}{2} \int_0^t |\Sigma(u)|^2 du \right\},$$

*for  $t \in [0, T]$ . Suppose that  $E_P(\varrho(T)) = 1$ , where  $E_P$  denotes expectation with respect to the probability measure  $P$ . Define the probability measure  $Q$  on  $(\Omega, \mathcal{F})$  by  $dQ := \varrho(T) dP$ .*

# Theorem 2 – Cont'd

---

*Then the process*

$$\widehat{W}(t) := W(t) - \int_0^t \Sigma(u) du, \quad t \in [0, T],$$

*is a standard Wiener process under the measure  $Q$ .*

# Backward Conditioning

---

Apply Girsanov's theorem with the process

$$\Sigma(u) := -\frac{\{h(u, S(u-a)) - r\}}{g(S(u-b))}, \quad u \in [0, T].$$

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The hypothesis on  $g$  implies that  $\Sigma$  is well-defined, since by Theorem 1,  $S(t) > 0$  for all  $t \in [0, T]$  a.s.. Clearly  $\Sigma(t)$ ,  $t \in [0, T]$ , is a predictable process.

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The process  $S(t)$ ,  $t \in [0, T]$ , is a.s. bounded because it is sample continuous. The hypothesis on  $g$  implies that  $1/g(v)$ ,  $v \in (0, \infty)$ , is bounded on bounded intervals.

Thus  $\int_0^T |\Sigma(u)|^2 du < \infty$  a.s..

# Backward Conditioning– Cont'd

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Remains to check the **integrability condition** in Girsanov's theorem.

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Set  $\mathcal{F}_t := \mathcal{F}_0$  for  $t \leq 0$ .



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Then  $\Sigma(u)$ ,  $u \in [0, T]$ , is measurable with respect to the  $\sigma$ -algebra  $\mathcal{F}_{T-l}$ .

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Then  $\Sigma(u)$ ,  $u \in [0, T]$ , is measurable with respect to the  $\sigma$ -algebra  $\mathcal{F}_{T-l}$ .

Hence, the stochastic integral  $\int_{T-l}^T \Sigma(u) dW(u)$  ***conditioned on*  $\mathcal{F}_{T-l}$**  has a **normal distribution** with mean zero and variance  $\int_{T-l}^T \Sigma(u)^2 du$ .

# Backward Conditioning– Cont'd

---

By normality (e.g. moment generating function):

$$\begin{aligned} E_P \left( \exp \left\{ \int_{T-l}^T \Sigma(u) dW(u) \right\} \middle| \mathcal{F}_{T-l} \right) \\ = \exp \left\{ \frac{1}{2} \int_{T-l}^T |\Sigma(u)|^2 du \right\} \end{aligned}$$

a.s..

# Backward Conditioning– Cont'd

By normality (e.g. moment generating function):

$$\begin{aligned} E_P \left( \exp \left\{ \int_{T-l}^T \Sigma(u) dW(u) \right\} \middle| \mathcal{F}_{T-l} \right) \\ = \exp \left\{ \frac{1}{2} \int_{T-l}^T |\Sigma(u)|^2 du \right\} \end{aligned}$$

a.s.. Hence

$$\begin{aligned} E_P \left( \exp \left\{ \int_{T-l}^T \Sigma(u) dW(u) - \frac{1}{2} \int_{T-l}^T |\Sigma(u)|^2 du \right\} \middle| \mathcal{F}_{T-l} \right) \\ = 1, \quad \text{a.s..} \end{aligned}$$

# Backward Conditioning– Cont'd

---

This implies:

$$\begin{aligned} E_P \left( \exp \left\{ \int_0^T \Sigma(u) dW(u) - \frac{1}{2} \int_0^T |\Sigma(u)|^2 du \right\} \middle| \mathcal{F}_{T-l} \right) \\ = \exp \left\{ \int_0^{T-l} \Sigma(u) dW(u) - \frac{1}{2} \int_0^{T-l} |\Sigma(u)|^2 du \right\} \end{aligned}$$

a.s..

# Backward Conditioning– Cont'd

Let  $k$  to be a positive integer such that  $0 \leq T - kl \leq l$ .  
Successive conditioning using backward steps of length  $l$ , and induction give:

$$\begin{aligned} E_P \left( \exp \left\{ \int_0^T \Sigma(u) dW(u) - \frac{1}{2} \int_0^T |\Sigma(u)|^2 du \right\} \middle| \mathcal{F}_{T-kl} \right) \\ = \exp \left\{ \int_0^{T-kl} \Sigma(u) dW(u) - \frac{1}{2} \int_0^{T-kl} |\Sigma(u)|^2 du \right\} \end{aligned}$$

a.s..

# Backward Conditioning– Cont'd

Take conditional expectation with respect to  $\mathcal{F}_0$  on both sides of above equation:

$$\begin{aligned} E_P \left( \exp \left\{ \int_0^T \Sigma(u) dW(u) \right. \right. \\ \left. \left. - \frac{1}{2} \int_0^T |\Sigma(u)|^2 du \right\} \middle| \mathcal{F}_0 \right) \\ = E_P \left( \exp \left\{ \int_0^{T-kl} \Sigma(u) dW(u) \right. \right. \\ \left. \left. - \frac{1}{2} \int_0^{T-kl} |\Sigma(u)|^2 du \right\} \middle| \mathcal{F}_0 \right) = 1 \end{aligned}$$

a.s..

# Backward Conditioning– Cont'd

---

Taking the expectation of the above equation, we get

$$E_P(\varrho(T)) = 1$$

where



# Backward Conditioning– Cont'd

Taking the expectation of the above equation, we get

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where

$$\varrho(T) := \exp \left\{ - \int_0^T \frac{\{h(u, S(u-a)) - r\}}{g(S(u-b))} dW(u) - \frac{1}{2} \int_0^T \left| \frac{h(u, S(u-a)) - r}{g(S(u-b))} \right|^2 du \right\}$$

a.s..

# Martingale Measure

Therefore, the Girsanov theorem (Theorem 2) applies and the process

$$\widehat{W}(t) := W(t) + \int_0^t \frac{\{h(u, S(u-a)) - r\}}{g(S(u-b))} du, \quad t \in [0, T],$$

is a standard Wiener process under the measure  $Q$  defined by:

$$dQ := \varrho(T) dP.$$

# Martingale Measure– Cont'd

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Since

$$\widehat{S}(t) = \int_0^t g(S(u - b)) d\widehat{W}(u), \quad t \in [0, T], \quad (4)$$

then  $\widehat{S}(t)$ ,  $t \in [0, T]$ , is a continuous  $Q$ -local martingale.

# Martingale Measure– Cont'd

---

Since

$$\widehat{S}(t) = \int_0^t g(S(u - b)) d\widehat{W}(u), \quad t \in [0, T], \quad (4)$$

then  $\widehat{S}(t)$ ,  $t \in [0, T]$ , is a continuous  $Q$ -local martingale.

By the representation

$$\widetilde{S}(t) = \varphi(0) + \int_0^t \widetilde{S}(u) d\widehat{S}(u), \quad t \in [0, T], \quad (3)$$

the discounted stock price  $\widetilde{S}(t)$ ,  $t \in [0, T]$ , is also a continuous  $Q$ -local martingale.

# No Arbitrage

---

I.e.  $Q$  is an equivalent local martingale measure.

# No Arbitrage

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By well-known results on trading strategies (e.g., Theorem 7.1 in [K.K]), it follows that the market consisting of  $\{B(t), S(t) : t \in [0, T]\}$  satisfies the **no-arbitrage property**:

# No Arbitrage

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I.e.  $Q$  is an **equivalent local martingale measure**.

By well-known results on trading strategies (e.g., Theorem 7.1 in [K.K]), it follows that the market consisting of  $\{B(t), S(t) : t \in [0, T]\}$  satisfies the **no-arbitrage property**: **There is no admissible self-financing strategy which gives an arbitrage opportunity.**

# Completeness

---

Next get **completeness** of the market  $\{B(t), S(t) : t \in [0, T]\}$ .



# Completeness

Next get **completeness** of the market  $\{B(t), S(t) : t \in [0, T]\}$ .

By proof of Theorem 1, the solution of the delayed stock model (1) satisfies:

$$S(t) = \varphi(0) \exp \left\{ \int_0^t g(S(u - b)) dW(u) + \int_0^t h(u, S(u - a)) du - \frac{1}{2} \int_0^t g(S(u - b))^2 du \right\}$$

a.s. for  $t \in [0, T]$ .

# Completeness– Cont'd

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Hence,

$$\begin{aligned} \tilde{S}(t) = & \varphi(0) \exp \left\{ \int_0^t g(S(u-b)) d\widehat{W}(u) \right. \\ & \left. - \frac{1}{2} \int_0^t g(S(u-b))^2 du \right\} \end{aligned} \quad (5)$$

for  $t \in [0, T]$ .

# Completeness– Cont'd

By definitions of  $\tilde{S}$ ,  $\widehat{W}$ ,  $\widehat{S}$  and equation (2), then for  $t \geq 0$ ,  $\mathcal{F}_t^S = \mathcal{F}_t^{\tilde{S}} = \mathcal{F}_t^{\widehat{W}} = \mathcal{F}_t^W$ , the  $\sigma$ -algebras generated by  $\{S(u) : u \leq t\}$ ,  $\{\tilde{S}(u) : u \leq t\}$ ,  $\{\widehat{W}(u) : u \leq t\}$ ,  $\{W(u) : u \leq t\}$ , respectively. (Clearly,  $\mathcal{F}_t^W \subseteq \mathcal{F}_t$ .)

# Completeness– Cont'd

By definitions of  $\tilde{S}$ ,  $\widehat{W}$ ,  $\widehat{S}$  and equation (2), then for  $t \geq 0$ ,  $\mathcal{F}_t^S = \mathcal{F}_t^{\tilde{S}} = \mathcal{F}_t^{\widehat{W}} = \mathcal{F}_t^W$ , the  $\sigma$ -algebras generated by  $\{S(u) : u \leq t\}$ ,  $\{\tilde{S}(u) : u \leq t\}$ ,  $\{\widehat{W}(u) : u \leq t\}$ ,  $\{W(u) : u \leq t\}$ , respectively. (Clearly,  $\mathcal{F}_t^W \subseteq \mathcal{F}_t$ .)

Let  $X$  be a **contingent claim**, viz. **an integrable non-negative  $\mathcal{F}_T^S$ -measurable random variable**. Consider the  $Q$ -martingale

$$M(t) := E_Q(e^{-rT}X \mid \mathcal{F}_t^S) = E_Q(e^{-rT}X \mid \mathcal{F}_t^{\widehat{W}}),$$

for  $t \in [0, T]$ .

# Completeness– Cont'd

By the martingale representation theorem, there exists an  $(\mathcal{F}_t^{\widehat{W}})$ -predictable process  $h_0(t)$ ,  $t \in [0, T]$ , such that

$$\int_0^T h_0(u)^2 du < \infty \quad a.s.,$$

and

$$M(t) = E_Q(e^{-rT} X) + \int_0^t h_0(u) d\widehat{W}(u), \quad t \in [0, T].$$

# Completeness– Cont'd

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Combining the two relations

$$d\tilde{S}(t) = \tilde{S}(t) d\hat{S}(t), \quad d\hat{S}(t) = g(S(t-b)) d\hat{W}(t),$$

gives:

$$d\tilde{S}(t) = \tilde{S}(t)g(S(t-b)) d\hat{W}(u), \quad t \in [0, T].$$

# Completeness– Cont'd

Combining the two relations

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gives:

$$d\tilde{S}(t) = \tilde{S}(t)g(S(t-b)) d\widehat{W}(u), \quad t \in [0, T].$$

Define

$$\pi_S(t) := \frac{h_0(t)}{\tilde{S}(t)g(S(t-b))}, \quad \pi_B(t) := M(t) - \pi_S(t)\tilde{S}(t)$$

for  $t \in [0, T]$ .

# Completeness– Cont'd

Consider the **strategy**  $\{(\pi_B(t), \pi_S(t)) : t \in [0, T]\}$  which consists of holding  $\pi_S(t)$  units of the stock and  $\pi_B(t)$  units of the bond at time  $t$ . The value of the **portfolio** at any time  $t \in [0, T]$  is:

$$V(t) := \pi_B(t)e^{rt} + \pi_S(t)S(t) = e^{rt}M(t).$$



# Completeness– Cont'd

Consider the **strategy**  $\{(\pi_B(t), \pi_S(t)) : t \in [0, T]\}$  which consists of holding  $\pi_S(t)$  units of the stock and  $\pi_B(t)$  units of the bond at time  $t$ . The value of the **portfolio** at any time  $t \in [0, T]$  is:

$$V(t) := \pi_B(t)e^{rt} + \pi_S(t)S(t) = e^{rt}M(t).$$

By the product rule and the definition of the strategy  $\{(\pi_B(t), \pi_S(t)) : t \in [0, T]\}$ , get

$$\begin{aligned} dV(t) &= e^{rt}dM(t) + M(t)d(e^{rt}) \\ &= \pi_B(t)d(e^{rt}) + \pi_S(t)dS(t), \end{aligned}$$

for  $t \in [0, T]$ .

# Completeness– Cont'd

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Hence,  $\{(\pi_B(t), \pi_S(t)) : t \in [0, T]\}$  is a **self-financing strategy**. Moreover,  $V(T) = e^{rT}M(T) = X$  a.s..

Therefore, the contingent claim  $X$  is **attainable**; thus the market  $\{B(t), S(t) : t \in [0, T]\}$  is **complete**: (**every contingent claim is attainable**).

# Completeness– Cont'd

Hence,  $\{(\pi_B(t), \pi_S(t)) : t \in [0, T]\}$  is a **self-financing strategy**. Moreover,  $V(T) = e^{rT} M(T) = X$  a.s..

Therefore, the contingent claim  $X$  is **attainable**; thus the market  $\{B(t), S(t) : t \in [0, T]\}$  is **complete**: (**every contingent claim is attainable**).

For the augmented market  $\{B(t), S(t), X : t \in [0, T]\}$  to satisfy the no-arbitrage property, the price of the claim  $X$  must be

$$V(t) = e^{-r(T-t)} E_Q(X \mid \mathcal{F}_t^S)$$

at each  $t \in [0, T]$  a.s. See, e.g., [B.R] or Theorem 9.2 in [K.K].

# Delayed Option Pricing Formula

---

Summarize above discussion in the following formula for the fair price  $V(t)$  of an option on the delayed stock.

# Theorem 3

---

*Suppose that the stock price  $S$  is given by the delayed stock model, where  $\varphi(0) > 0$  and  $g$  satisfies the given hypotheses. Let  $T$  be the maturity time of an option (contingent claim) on the stock with payoff function  $X$ , i.e.,  $X$  is an  $\mathcal{F}_T^S$ -measurable non-negative integrable random variable. Then at any time  $t \in [0, T]$ , the fair price  $V(t)$  of the option is given by the formula*

$$V(t) = e^{-r(T-t)} E_Q(X \mid \mathcal{F}_t^S), \quad (6)$$

# Theorem 3 – Cont'd

where  $Q$  denotes the probability measure on  $(\Omega, \mathcal{F})$  defined by  $dQ := \varrho(T) dP$  with

$$\varrho(t) := \exp \left\{ - \int_0^t \frac{\{h(u, S(u-a)) - r\}}{g(S(u-b))} dW(u) - \frac{1}{2} \int_0^t \left| \frac{h(u, S(u-a)) - r}{g(S(u-b))} \right|^2 du \right\}$$

for  $t \in [0, T]$ .

# Theorem 3 – Cont'd

*The measure  $Q$  is a local martingale measure and the market is complete.*

*Moreover, there is an adapted and square integrable process  $h_0(u)$ ,  $u \in [0, T]$  such that*

$$E_Q(e^{-rT}X \mid \mathcal{F}_t^S) = E_Q(e^{-rT}X) + \int_0^t h_0(u) d\widehat{W}(u),$$

*for  $t \in [0, T]$ , where  $\widehat{W}$  is a standard  $Q$ -Wiener process given by*

# Theorem 3 – Cont'd

$$\widehat{W}(t) := W(t) + \int_0^t \frac{\{h(u, S(u-a)) - r\}}{g(S(u-b))} du, \quad t \in [0, T],$$

*The hedging strategy is given by*

$$\pi_S(t) := \frac{h_0(t)}{\widetilde{S}(t)g(S(t-b))}, \quad (7)$$

$$\pi_B(t) := M(t) - \pi_S(t)\widetilde{S}(t),$$

*for  $t \in [0, T]$ .*



# Delayed B-S Formula

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The following result is a consequence of Theorem 3. It gives a Black-Scholes-type formula for the value of a European option on the stock at times prior to maturity.

# Delayed B-S Formula

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The following result is a consequence of Theorem 3. It gives a Black-Scholes-type formula for the value of a European option on the stock at times prior to maturity.

Formula is **explicit** during **last delay period** before maturity, or when delay is **larger** than maturity interval.

# Theorem 4

*Assume the conditions of Theorem 3. Let  $V(t)$  be the fair price of a European call option written on the stock  $S$  with exercise price  $K$  and maturity time  $T$ . Let  $\varphi$  denote the standard normal distribution function:*

$$\varphi(x) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-u^2/2} du, \quad x \in \mathbf{R}.$$

*Then for all  $t \in [T - l, T]$  (where  $l := \min\{a, b\}$ ),  $V(t)$  is given by*

$$V(t) = S(t)\varphi(\beta_+(t)) - Ke^{-r(T-t)}\varphi(\beta_-(t)), \quad (8)$$

# Theorem 4 – Cont'd

where

$$\beta_{\pm}(t) := \frac{\log \frac{S(t)}{K} + \int_t^T \left( r \pm \frac{1}{2} g(S(u-b))^2 \right) du}{\sqrt{\int_t^T g(S(u-b))^2 du}}.$$

If  $T > l$  and  $t < T - l$ , then

$$V(t) = e^{rt} E_Q \left( H \left( \tilde{S}(T-l), -\frac{1}{2} \int_{T-l}^T g(S(u-b))^2 du, \int_{T-l}^T g(S(u-b))^2 du \right) \middle| \mathcal{F}_t \right) \quad (9)$$

# Theorem 4 – Cont'd

where  $H$  is given by

$$H(x, m, \sigma^2) := xe^{m+\sigma^2/2}\varphi(\alpha_1(x, m, \sigma)) \\ - Ke^{-rT}\varphi(\alpha_2(x, m, \sigma)),$$

and

$$\alpha_1(x, m, \sigma) := \frac{1}{\sigma} \left[ \log \left( \frac{x}{K} \right) + rT + m + \sigma^2 \right],$$

$$\alpha_2(x, m, \sigma) := \frac{1}{\sigma} \left[ \log \left( \frac{x}{K} \right) + rT + m \right],$$

for  $\sigma, x \in \mathbf{R}^+$ ,  $m \in \mathbf{R}$ .

# Theorem 4 – Cont'd

---

*The hedging strategy is given by*

$$\begin{aligned}\pi_S(t) &= \varphi(\beta_+(t)), \\ \pi_B(t) &= -Ke^{-rT}\varphi(\beta_-(t)),\end{aligned}\tag{10}$$

*for  $t \in [T - \ell, T]$ .*

# Remarks 2

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If  $g(x) = 1$  for all  $x \in \mathbf{R}^+$  then equation (8) reduces to the classical Black and Scholes formula.

# Remarks 2

---

If  $g(x) = 1$  for all  $x \in \mathbf{R}^+$  then equation (8) reduces to the classical Black and Scholes formula.

In contrast with the classical (non-delayed) Black and Scholes formula, the fair price  $V(t)$  in the delayed model in Theorem 4 depends not only on the stock price  $S(t)$  at the present time  $t$ , but also on the whole segment  $\{S(v) : v \in [t - b, T - b]\}$ . ( $[t - b, T - b] \subset [0, t]$  since  $t \geq T - l$  and  $l \leq b$ .)



# Proof of Theorem 4

---

Consider a European call option in the above market with exercise price  $K$  and maturity time  $T$ . Taking  $X = (S(T) - K)^+$  in Theorem 3, the fair price  $V(t)$  of the option is given by

$$\begin{aligned} V(t) &= e^{-r(T-t)} E_Q((S(T) - K)^+ | \mathcal{F}_t) \\ &= e^{rt} E_Q((\tilde{S}(T) - Ke^{-rT})^+ | \mathcal{F}_t), \end{aligned} \tag{11}$$

at any time  $t \in [0, T]$ .

# Proof of Theorem 4 – Cont'd

We now derive an explicit formula for the option price  $V(t)$  at any time  $t \in [T - l, T]$ . The representation (5) of  $\tilde{S}(t)$  implies:

$$\tilde{S}(T) = \tilde{S}(t) \exp \left\{ \int_t^T g(S(u - b)) d\widehat{W}(u) - \frac{1}{2} \int_t^T g(S(u - b))^2 du \right\}$$

for all  $t \in [0, T]$ . Clearly  $\tilde{S}(t)$  is  $\mathcal{F}_t$ -measurable. If  $t \in [T - l, T]$ , then  $-\frac{1}{2} \int_t^T g(S(u - b))^2 du$  is also  $\mathcal{F}_t$ -measurable.

# Proof of Theorem 4 – Cont'd

Furthermore, when conditioned on  $\mathcal{F}_t$ , the distribution of  $\int_t^T g(S(u - b)) d\widehat{W}(u)$  under  $Q$  is the same as that of  $\sigma\xi$ , where  $\xi$  is a Gaussian  $N(0, 1)$ -distributed random variable, and  $\sigma^2 = \int_t^T g(S(u - b))^2 du$ . Consequently, the fair price at time  $t$  is given by

$$V(t) = e^{rt} H \left( \tilde{S}(t), -\frac{1}{2} \int_t^T g(S(u - b))^2 du, \int_t^T g(S(u - b))^2 du \right),$$

# Proof of Theorem 4 – Cont'd

---

where

$$H(x, m, \sigma^2) := E_Q(xe^{m+\sigma\xi} - Ke^{-rT})^+,$$

for  $\sigma, x \in \mathbf{R}^+$ ,  $m \in \mathbf{R}$ . Now, an elementary computation yields the following:

$$\begin{aligned} H(x, m, \sigma^2) = & xe^{m+\sigma^2/2}\varphi(\alpha_1(x, m, \sigma)) \\ & - Ke^{-rT}\varphi(\alpha_2(x, m, \sigma)). \end{aligned}$$

# Proof of Theorem 4 – Cont'd

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Therefore,  $V(t)$  takes the form:

$$V(t) = S(t)\varphi(\beta_+) - Ke^{-r(T-t)}\varphi(\beta_-), \quad (12)$$

where

$$\beta_{\pm} = \frac{\log \frac{S(t)}{K} + \int_t^T \left( r \pm \frac{1}{2}g(S(u-b))^2 \right) du}{\sqrt{\int_t^T g(S(u-b))^2 du}}.$$

# Proof of Theorem 4 – Cont'd

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For  $T > l$  and  $t < T - l$ , from the representation (5) of  $\tilde{S}(t)$ , we have

$$\tilde{S}(T) = \tilde{S}(T - l) \exp \left\{ \int_{T-l}^T g(S(u - b)) d\widehat{W}(u) - \frac{1}{2} \int_{T-l}^T g(S(u - b))^2 du \right\}.$$

# Proof of Theorem 4 – Cont'd

---

Consequently, the option price at time  $t$  with  $t < T - l$  is given by

$$V(t) = e^{rt} E_Q \left( H \left( \tilde{S}(T - l), -\frac{1}{2} \int_{T-l}^T g(S(u - b))^2 du, \int_{T-l}^T g(S(u - b))^2 du \right) \middle| \mathcal{F}_t \right).$$

# Proof of Theorem 4 – Cont'd

Consequently, the option price at time  $t$  with  $t < T - l$  is given by

$$V(t) = e^{rt} E_Q \left( H \left( \tilde{S}(T - l), -\frac{1}{2} \int_{T-l}^T g(S(u - b))^2 du, \int_{T-l}^T g(S(u - b))^2 du \right) \middle| \mathcal{F}_t \right).$$

To calculate the hedging strategy for  $t \in [T - \ell, T]$ , it suffices to use an idea from [B.R], pages 95–96. This completes the proof of the theorem.  $\diamond$



## Remark 3

During **last delay period**  $[T - l, T]$ , it is possible to rewrite the option price  $V(t)$ ,  $t \in [T - l, T]$  in terms of the solution of a random Black-Scholes pde of the form

$$\left. \begin{aligned} \frac{\partial F(t, x)}{\partial t} &= -\frac{1}{2}g(S(t - b))^2 x^2 \frac{\partial^2 F(t, x)}{\partial x^2} - rx \frac{\partial F(t, x)}{\partial x} \\ &\quad + rF(t, x), \quad 0 < t < T \\ F(T, x) &= (x - K)^+, \quad x > 0. \end{aligned} \right\} \quad (13)$$

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Above time-dependent random final-value problem admits a unique  $(\mathcal{F}_t)_{t \geq 0}$ -adapted random field  $F(t, x)$ .

## Remark 3 – Cont'd

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Using the classical Itô-Ventzell formula ([Kun]) and (6) of Theorem 3, it can be shown that

$$V(t) = e^{-r(T-t)} F(t, S(t)), \quad t \in [T - b, T].$$

## Remark 3 – Cont'd

---

Using the classical Itô-Ventzell formula ([Kun]) and (6) of Theorem 3, it can be shown that

$$V(t) = e^{-r(T-t)} F(t, S(t)), \quad t \in [T - b, T].$$

Note that the above representation is no longer valid if  $t \leq T - b$ , because in this range, the solution  $F$  of the final-value problem (9) is *anticipating* with respect to the filtration  $(\mathcal{F}_t)_{t \geq 0}$ .

# A Stock Model with Variable Delay

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Consider an alternative model for the stock price dynamics with variable delay.

# A Stock Model with Variable Delay

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Throughout this section, suppose  $h$  is a given fixed positive number. Denote  $\lfloor t \rfloor := kh$  if  $kh \leq t < (k+1)h$ .

# A Stock Model with Variable Delay

Consider an alternative model for the stock price dynamics with variable delay.

Throughout this section, suppose  $h$  is a given fixed positive number. Denote  $\lfloor t \rfloor := kh$  if  $kh \leq t < (k+1)h$ . Suppose market consist of a **riskless asset**  $\xi$  with a variable (deterministic) continuous rate of return  $\lambda$ , and a **stock**  $S$  satisfying sdde

$$\left. \begin{aligned} d\xi(t) &= \lambda(t)\xi(t) dt \\ dS(t) &= f(t, S(\lfloor t \rfloor))S(t)dt + g(t, S(\lfloor t \rfloor))S(t)dW(t) \end{aligned} \right\} \quad (14)$$

for  $t \in (0, T]$ .

# A Model with Variable Delay – Cont'd

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Initial conditions  $\xi(0) = 1$  and  $S(0) > 0$ .



# A Model with Variable Delay – Cont'd

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$(\mathcal{F}_t)_{0 \leq t \leq T}$  and  $W$  are as before.

# A Model with Variable Delay – Cont'd

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Initial conditions  $\xi(0) = 1$  and  $S(0) > 0$ .

$(\mathcal{F}_t)_{0 \leq t \leq T}$  and  $W$  are as before.

$f : [0, T] \times \mathbf{R} \rightarrow \mathbf{R}$  is continuous.

# A Model with Variable Delay – Cont'd

---

Initial conditions  $\xi(0) = 1$  and  $S(0) > 0$ .

$(\mathcal{F}_t)_{0 \leq t \leq T}$  and  $W$  are as before.

$f : [0, T] \times \mathbf{R} \rightarrow \mathbf{R}$  is continuous.

$g : [0, T] \times \mathbf{R} \rightarrow \mathbf{R}$  is continuous.

# A Model with Variable Delay – Cont'd

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**Initial conditions**  $\xi(0) = 1$  and  $S(0) > 0$ .

$(\mathcal{F}_t)_{0 \leq t \leq T}$  and  $W$  are as before.

$f : [0, T] \times \mathbf{R} \rightarrow \mathbf{R}$  is continuous.

$g : [0, T] \times \mathbf{R} \rightarrow \mathbf{R}$  is continuous.

$g(t, v) \neq 0$  for all  $(t, v) \in [0, T] \times \mathbf{R}$ .

# A Model with Variable Delay – Cont'd

---

Initial conditions  $\xi(0) = 1$  and  $S(0) > 0$ .

$(\mathcal{F}_t)_{0 \leq t \leq T}$  and  $W$  are as before.

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The model is **feasible**: That is  $S(t) > 0$  a.s. for all  $t > 0$ .

# A Model with Variable Delay – Cont'd

---

Initial conditions  $\xi(0) = 1$  and  $S(0) > 0$ .

$(\mathcal{F}_t)_{0 \leq t \leq T}$  and  $W$  are as before.

$f : [0, T] \times \mathbf{R} \rightarrow \mathbf{R}$  is continuous.

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$g(t, v) \neq 0$  for all  $(t, v) \in [0, T] \times \mathbf{R}$ .

The model is **feasible**: That is  $S(t) > 0$  a.s. for all  $t > 0$ .

Follows by an argument similar to the proof of Theorem 1.

# Theorem 5

*Suppose that the stock price  $S$  is given by the sdde (14), where  $S(0) > 0$  and  $f, g$  satisfy given hypotheses. Let  $T$  be the maturity time of an option (contingent claim) on the stock with payoff function  $X$ , i.e.,  $X$  is an  $\mathcal{F}_T^S$ -measurable non-negative integrable random variable. Then at any time  $t \in [0, T]$ , the fair price  $V(t)$  of the option is given by the formula*

$$V(t) = E_Q(X \mid \mathcal{F}_t^S) e^{-\int_t^T \lambda(s) ds}, \quad (15)$$

*where  $Q$  denotes the probability measure on  $(\Omega, \mathcal{F})$  defined by  $dQ := \varrho(T) dP$  with*

# Theorem 5 – Cont'd

$$\varrho(t) := \exp \left\{ - \int_0^t \frac{\{f(u, S(\lfloor u \rfloor)) - \lambda(u)\}}{g(u, S(\lfloor u \rfloor))} dW(u) - \frac{1}{2} \int_0^t \left| \frac{f(u, S(\lfloor u \rfloor)) - \lambda(u)}{g(u, S(\lfloor u \rfloor))} \right|^2 du \right\}$$

*for  $t \in [0, T]$ . The measure  $Q$  is a local martingale measure and the market is complete.*



# Theorem 5 – Cont'd

*Moreover, there is an adapted and square integrable process  $h_1(t)$ ,  $t \in [0, T]$ , such that*

$$E_Q \left( \frac{X}{\xi(T)} \middle| \mathcal{F}_t^S \right) = E_Q \left( \frac{X}{\xi(T)} \right) + \int_0^t h_1(u) d\widehat{W}(u),$$
$$t \in [0, T],$$

*where*

$$\widehat{W}(t) := W(t) + \int_0^t \frac{\{f(u, S(\lfloor u \rfloor)) - \lambda(u)\}}{g(u, S(\lfloor u \rfloor))} du, \quad t \in [0, T].$$

# Theorem 5 – Cont'd

---

*The hedging strategy is given by*

$$\pi_S(t) := \frac{h_1(t)}{\widetilde{S}(t)g(t, S(\lfloor t \rfloor))}, \quad (16)$$

$$\pi_\xi(t) := M(t) - \pi_S(t)\widetilde{S}(t),$$

*for  $t \in [0, T]$ .*

# Theorem 5 – Cont'd

---

*The hedging strategy is given by*

$$\begin{aligned}\pi_S(t) &:= \frac{h_1(t)}{\tilde{S}(t)g(t, S(\lfloor t \rfloor))}, \\ \pi_\xi(t) &:= M(t) - \pi_S(t)\tilde{S}(t),\end{aligned}\tag{16}$$

*for  $t \in [0, T]$ .*

The following result gives a Black-Scholes-type formula for the value of a European option on the stock at any time prior to maturity.

# Theorem 6

*Assume the conditions of Theorem 5. Let  $V(t)$  be the fair price of a European call option written on the stock  $S$  with exercise price  $K$  and maturity time  $T$ . Then for all  $t \in [T - \lfloor T \rfloor, T]$ ,  $V(t)$  is given by*

$$V(t) = S(t)\varphi(\beta_+(t)) - K\varphi(\beta_-(t))e^{-\int_t^T \lambda(s)ds}, \quad (17)$$

*where*

$$\beta_{\pm}(t) := \frac{\log \frac{S(t)}{K} + \int_t^T (\lambda(u) \pm \frac{1}{2}g(u, S(\lfloor u \rfloor))^2) du}{\sqrt{\int_t^T g(u, S(\lfloor u \rfloor))^2 du}}.$$

# Theorem 6 – Cont'd

*If  $T > h$  and  $t < T - \lfloor T \rfloor$ , then*

$$\begin{aligned} V(t) = & e^{\int_0^t \lambda(s) ds} E_Q \left( H \left( \tilde{S}(T - \lfloor T \rfloor), \right. \right. \\ & - \frac{1}{2} \int_{T - \lfloor T \rfloor}^T g(u, S(\lfloor u \rfloor))^2 du, \\ & \left. \left. \int_{T - \lfloor T \rfloor}^T g(u, S(\lfloor u \rfloor))^2 du \right) \middle| \mathcal{F}_t \right) \end{aligned} \quad (18)$$

*where  $H$  is given by*

# Theorem 6 – Cont'd

$$H(x, m, \sigma^2) := xe^{m+\sigma^2/2}\varphi(\alpha_1(x, m, \sigma)) \\ - K\varphi(\alpha_2(x, m, \sigma))e^{-\int_0^T \lambda(s)ds},$$

*and*

$$\alpha_1(x, m, \sigma) := \frac{1}{\sigma} \left[ \log \left( \frac{x}{K} \right) + \int_0^T \lambda(s)ds + m + \sigma^2 \right],$$

$$\alpha_2(x, m, \sigma) := \frac{1}{\sigma} \left[ \log \left( \frac{x}{K} \right) + \int_0^T \lambda(s)ds + m \right],$$

*for*  $\sigma, x \in \mathbf{R}^+, m \in \mathbf{R}$ .

# Theorem 6 – Cont'd

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*The hedging strategy is given by*

$$\pi_S(t) = \varphi(\beta_+(t)),$$

$$\pi_\xi(t) = -K\varphi(\beta_-(t))e^{-\int_0^T \lambda(s)ds},$$

*for  $t \in [T - \lfloor T \rfloor, T]$ .*

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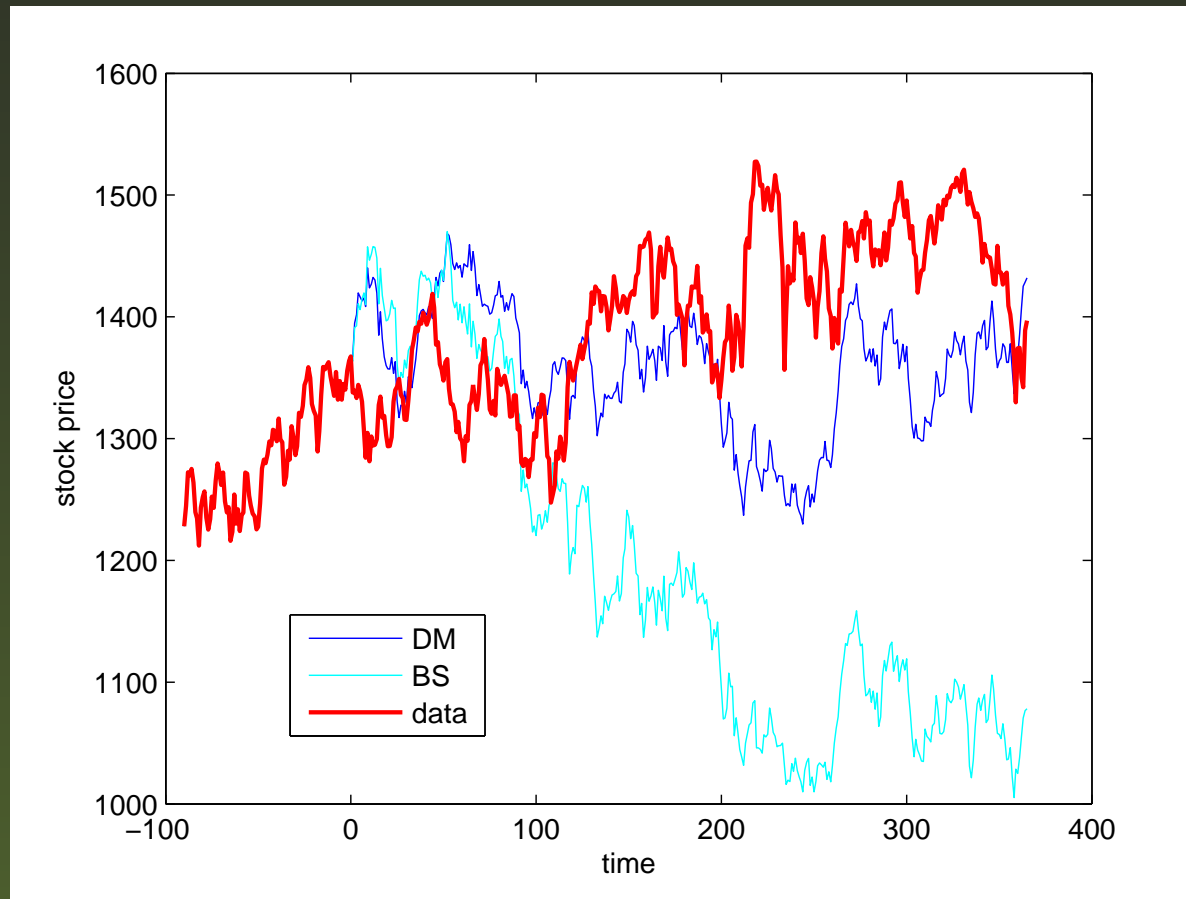
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# Stock Dynamics-Simulation

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# Stock Dynamics-Simulation



Stock prices when  $h = \text{constant}$ ,  $b = 2$ ,  $T = 365$   
Stock data: DJX Index at CBOE.

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THANK YOU!